# Minimizing Symmetric Convex Functions over Hybrid of Continuous and Discrete Convex Sets

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## Abstract

We study the problem of minimizing a given symmetric strictly convex function over the Minkowski sum of an integral base-polyhedron and an M-convex set. This problem has a hybrid of continuous and discrete structures. This emerges from the problem of allocating mixed goods, consisting of both divisible and indivisible goods, to agents with binary valuations so that the fairness measure, such as the Nash welfare, is maximized. It is known that both an integral base-polyhedron and an M-convex set have similar and nice properties, and the non-hybrid case can be solved in polynomial time. While the hybrid case lacks some of these properties, we show the structure of an optimal solution. Moreover, we exploit a proximity inherent in the problem. Through our findings, we demonstrate that our problem is NP-hard even in the fair allocation setting where all indivisible goods are identical. Moreover, we provide a polynomial-time algorithm for the fair allocation problem when all divisible goods are identical.

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## 1 Introduction

In this paper, we study the hybrid problem of discrete and continuous structures. We are given a finite set $N$, a symmetric strictly convex function $\Phi\colon \mathbb{R}^N \to \mathbb{R}$ and two supermodular functions $f_C, f_M\colon 2^N \to \mathbb{Z}_+$. We assume that function values can be accessed by an oracle. For a supermodular function $f$, we call

$$\overline{\mathbf{B}} = \left\{x \in \mathbb{R}^N : x(N) = f(N) \quad \text{and} \quad x(X) \geq f(X) \quad (\forall X \subseteq N)\right\}$$

the *integral base-polyhedron* of $f$, and $\ddot{\mathbf{B}} = \overline{\mathbf{B}} \cap \mathbb{Z}^N$ the *M-convex* set. Let $\overline{\mathbf{B}}_C$, $\overline{\mathbf{B}}_M$ be the integral base-polyhedra of $f_C$, $f_M$, respectively, and let $\ddot{\mathbf{B}}_M = \overline{\mathbf{B}}_M \cap \mathbb{Z}^N$. In addition, let


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$\mathbf{B}_E$ be the Minkowski sum of $\ddot{\mathbf{B}}_M$ and $\overline{\mathbf{B}}_C$, i.e., $\mathbf{B}_E = \{x + y : x \in \ddot{\mathbf{B}}_M,\ y \in \overline{\mathbf{B}}_C\}$. The goal of the problem is to find a vector $z$ that attains

$$\min_{z\in\mathbf{B}_E} \Phi(z)\ (= \min_{x\in\ddot{\mathbf{B}}_M} \min_{y\in\overline{\mathbf{B}}_C} \Phi(x + y)). \tag{1}$$

An optimal solution of this problem is called $\Phi$-*minimizer* (on $\mathbf{B}_E$).

When $f_M = 0$ (i.e., $\mathbf{B}_E = \overline{\mathbf{B}}_C$), it is known that an integral base-polyhedron has a common unique minimizer independent of $\Phi$, and the minimizer can be characterized by a structure called the *principal partition* [18, 34] (see Section 2.2 for the definition and details). By this structure, the problem (1) can be solved in polynomial time [38]. When $f_C = 0$, it is known that a $\Phi$-minimizer on an M-convex set can be characterized by the *canonical partition* [15], which is an aggregation of the principal partition. Additionally, the set of $\Phi$-minimizers does not depend on $\Phi$ [15] and a $\Phi$-minimizer can be found in polynomial time [16]. Furthermore, a *proximity theorem* has been established [17]. This theorem states that a $\Phi$-minimizer in an M-convex set lies within a unit hypercube that contains the $\Phi$-minimizer in the corresponding integral base-polyhedron.

The hybrid problem (1) appears in the fair allocation of a mix of divisible and indivisible goods, which has recently been attracting attention [5,7,10,31,32,33,40]. Let $N = \{1, 2, \dots, n\}$ be the set of agents. Let $C$ and $M$ be the sets of divisible and indivisible goods, respectively, and let also $E = C \cup M$. Each agent $i$ has a binary valuation $v_{ie} \in \{0, 1\}$ for each good $e$. An allocation is a matrix $\pi \in [0, 1]^{N\times E}$ such that $\pi_{ie} \in \{0, 1\}$ for all $i \in N$ and $e \in M$. The entry $\pi_{ie}$ means the allocated amount of good $e$ to agent $i$. Throughout this paper, we only consider utilitarian optimum allocations, that is, $\pi_{ie} > 0$ only if $v_{ie} = 1$. Agents have additive utility, and the utility of agent $i$ in allocation $\pi$ is $\pi_i(E) = \sum_{e\in E} v_{ie}\pi_{ie}$. For an allocation $\pi$, a vector $z = (\pi_1(E), \dots, \pi_n(E))$ is called a utility vector of $\pi$.

The problem of finding a utility vector with the maximum *Nash welfare* (MNW), which is a prominent fairness measure, can be reduced to our problem (1). Roughly speaking, the set of possible utility vectors by divisible goods $C$ forms an integral base-polyhedron $\overline{\mathbf{B}}_C$, and the set by indivisible goods $M$ forms an *M-convex set* $\ddot{\mathbf{B}}_M$, which is the set of integral vectors in an integral base-polyhedron. Maximizing the Nash welfare corresponds to minimizing a symmetric strictly convex function $\Phi(z) = -\sum_{i\in N} \log(z_i + \varepsilon)$ for sufficiently small $\varepsilon > 0$ (depending on the instance). Another standard fairness measure called *egalitarian social welfare* (max-min fairness) also can be represented by a symmetric strictly convex function. For a given symmetric strictly convex function $\Phi$, we call an allocation $\pi$ $\Phi$-*fair* if its utility vector $z = (\pi_1(E), \dots, \pi_n(E))$ is a $\Phi$-minimizer. We will detail these in Section 2.

Unfortunately, the hybrid case does not inherit nice properties of continuous or discrete cases even in the fair allocation case. The set $\mathbf{B}_E$ is not necessarily an integral base-polyhedron or an M-convex set. It also does not work to find allocations of divisible and indivisible goods separately and combine them. We can observe these from the following example.

► **Example 1.** Suppose that there are one indivisible good $g$, one divisible good $c$, and three agents who desire both goods. Let $\Phi(z) = -z_1 \cdot z_2 \cdot z_3$. In this case, allocating $c$ equally to the three agents minimizes $\Phi$ when considering only $c$. However, allocating $g$ to agent 1 and $c$ to agents 2 and 3 equally minimizes $\Phi$ for mixed goods. In addition, the set of possible utility vectors is not an M-convex set since it contains fractional utility vectors and not an integral base-polyhedron since it is not convex (see Figure 1).

In addition, we will see that the uniqueness of a $\Phi$-minimizer set no longer holds (Example 7). Therefore, existing results are not applicable to our problem.

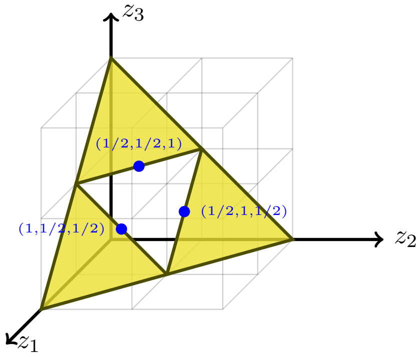


**Figure 1** The set of possible utility vectors in Example 1. The blue points are minimizers of $\Phi$.

## 1.1 Our contribution

First, we investigate the structure of the problem (1). Fortunately, we show that the hybrid problem still retains a structure of the canonical partition (Lemma 28), which is originally defined for the discrete case [15]. Namely, there exists integers $\beta_1 > \cdots > \beta_q$ and a partition $N_1, \dots, N_q$ of $N$ such that $\beta_j - 1 \leq z_i^* \leq \beta_j$ for any $\Phi$-minimizer $z^*$, each $j = 1, \dots, q$ and $i \in N_j$. The proof is based on exchanging element values of a solution. Since existing exchange properties in the discrete case are insufficient to deal with the hybrid problem, we need to introduce new exchange properties. Moreover, we discuss an optimality criterion in terms of an exchange graph, and unlike the discrete case, elaborate analysis of the graph is required. By this result, we can see that an optimal integral solution (i.e., $\arg\min_{z \in \ddot{\mathbf{B}}_M + \ddot{\mathbf{B}}_C} \Phi(z)$) is a good approximation solution for (1) in the sense that the $\ell_\infty$ distance from the optimal solution is at most 1. We remark that the canonical partition together can be found in polynomial time, and an optimal integral solution can be easily obtained from it.

In addition, by using the canonical partition, we can demonstrate a proximity theorem (Theorem 2). Namely, $\Phi$-minimizer on $\mathbf{B}_E$ lies within a unit box containing the $\Phi$-minimizer on an integral base-polyhedron $\overline{\mathbf{B}}_E$ of $f_E = f_M + f_C$. This generalizes a proximity theorem for the discrete case [17].

▶ **Theorem 2.** *Let $\Phi$ be a symmetric strictly convex function. For any $z^* \in \arg\min_{z \in \mathbf{B}_E} \Phi(z)$ and $\bar{z} \in \arg\min_{z \in \overline{\mathbf{B}}_E} \Phi(z)$, we have $\lfloor \bar{z}_i \rfloor \leq z_i^* \leq \lceil \bar{z}_i \rceil$ for all $i \in N$.*

Second, by applying the above results, we analyze the computational complexity of the problem (1) where $\overline{\mathbf{B}}_C$ and $\ddot{\mathbf{B}}_M$ arise from fair allocation. As a negative result, we show that the problem is NP-hard even when indivisible goods are *identical*, i.e., for each agent $i$, either $v_{ie} = 1$ ($\forall e \in M$) or $v_{ie} = 0$ ($\forall e \in M$).

▶ **Theorem 3.** *For any fixed symmetric strictly convex function $\Phi$, finding a $\Phi$-fair allocation is NP-hard even when indivisible goods are identical.*

We also prove that computing an MNW allocation and an optimal egalitarian allocation are both NP-hard. These results highlight the difficulty of the mixed goods case because the problems can be solved in polynomial time when there are only divisible goods or only indivisible goods.

As a positive result, we show the following tractability when divisible goods are identical. This class includes the case when the divisible goods are money.

▶ **Theorem 4.** *Let $\Phi$ be a symmetric strictly convex function. There exists a polynomial-time algorithm that finds a $\Phi$-fair allocation if all the divisible goods are identical.*

This result may be interesting, because in fact, finding an allocation that attains a given utility vector is NP-hard (see Appendix B in the full version [29]). Nevertheless, Theorem 4 says that we can obtain not only a $\Phi$-minimizer (utility vector) but also an allocation that attains the utility vector.

A key tool to construct our algorithm is the canonical partition for the mixed goods. By applying it, we can partition goods as $E_1, \ldots, E_q$ and agents as $N_1, \ldots, N_q$ so that goods in $E_j$ are allocated to agents in $N_j$ in a $\Phi$-fair allocation (Theorem 29). Thanks to this structure, a $\Phi$-minimizer can be found by independently solving the subproblems of assigning $E_j$ to $N_j$ for $j = 1, 2, \ldots, q$. In each subproblem, the utility of every agent is almost the same. However, unlike the continuous or discrete case, an optimal allocation depends on $\Phi$ (see Examples 6 and 7). Thus, it is not easy to obtain a full characterization of minimizers.

Due to the space limitation, some proofs are omitted. They can be found in the full version [29].

## 1.2 Related work

The (integral) base-polyhedron has been studied in the theory of matroids and submodular functions [19]. The concept of M-convex sets [36] is defined as a set of integral vectors satisfying certain exchange axioms. *Discrete convex analysis* [37] is a framework of convex analysis in discrete settings, including M-convexity.

The concepts of continuous/discrete hybrid convexity have been proposed previously [35, 44]. In particular, an optimality criterion for an integral polyhedral hybrid M-convex function minimization is known [35]. However, the functions treated in the present paper are hybrid M-convex functions that are not necessarily integral polyhedral.

For the fair allocation of divisible homogeneous goods with additive valuations (not restricted to binary), an MNW allocation corresponds to a market equilibrium of a special case of the Fisher's market model (see, e.g., [39]). Moreover, an MNW allocation is *envy-free* (EF) [43, 45], that is, no agent envies any other agent. It is known that this problem can be solved in strongly polynomial time [41, 46].

For the fair allocation of indivisible goods with additive valuations, Caragiannis et al. [10] proved that an MNW allocation is *envy-free up to one good* (EF1), that is, each agent $i$ does not envy another agent $j$ if some indivisible good is removed from the bundle of agent $j$. Since computing an MNW allocation is hard in general [30], there is a series of research to design an approximation algorithm [1, 11, 12, 13, 22]. Benabbou et al. [6] proved that the set of MNW allocations coincides with that of minimizers of any symmetric strictly convex function, even when the utility of each agent is represented by a matroid rank function.[1] Harvey et al. [26] proposed efficient algorithms for computing an allocation that minimizes a certain symmetric strictly convex function. When agents have binary additive valuations, an MNW allocation can be computed in polynomial time [4, 14]. Truthful mechanisms to find an MNW allocation are also proposed [2, 24].

Fair allocation with a mixture of divisible and indivisible goods has recently gained attention. Bei et al. [5] introduced a fairness notion called *envy-freeness for mixed goods* (EFM) as a generalization of EF and EF1 notions. Very recently, Li et al. [31] proposed a truthful mechanism that outputs an EFM allocation for the case where agents have binary

[1] Note that Theorem 2 is an extension of this result. Specifically, we construct an "augmenting" path of [6, Section 3.2] for a hybrid situation.

additive valuations on indivisible goods and a common valuation on a single divisible good (e.g., money). They also showed that their mechanism runs in polynomial time, and its output achieves MNW. We remark that their algorithm does not work in our problem even when divisible goods are identical because we allow some agents to have value 0 on them. In addition, fair allocation of indivisible goods with subsidy [3, 8, 9, 23, 25, 28] is related to the problem since subsidy could be viewed as a divisible good. For more details, see a survey paper by Liu et al. [32].

## 2 Preliminaries

In this section, we explain the relationship between fair allocation of mixed goods and the hybrid problem (1). Then we introduce the canonical partition for the discrete case.

For $k \in \mathbb{N}$, we denote $[k] = \{1, 2, \dots, k\}$. Let $N = [n]$ be a finite set. A set function $f$ over $N$ is called *supermodular* if

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y) \quad (\forall X, Y \subseteq N)$$

and *submodular* if $-f$ is supermodular. For a subset $X \subseteq N$ and a vector $x \in \mathbb{R}^N$, we denote $x(X) = \sum_{i \in X} x_i$. For an integer-valued supermodular set function $f$ on $N$ for which $f(\emptyset) = 0$ (normalized), the *integral base-polyhedron* $\overline{\mathbf{B}}$ of $f$ is defined as

$$\overline{\mathbf{B}} = \left\{x \in \mathbb{R}^N : x(N) = f(N) \text{ and } x(X) \geq f(X) \ (\forall X \subseteq N)\right\}.$$

In addition, we call the set $\ddot{\mathbf{B}}$ of the integer vectors in an integral base-polyhedron $\overline{\mathbf{B}}$ an *M-convex set*. Note that an M-convex set $\ddot{\mathbf{B}}$ induces an integral base-polyhedron $\overline{\mathbf{B}}$ as its convex hull.

We say that a function $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ is *symmetric* if

$$\Phi(z_1, z_2, \dots, z_n) = \Phi(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)})$$

for all permutations $\sigma$ over $[n]$. We say that a function $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ is strictly *convex* if

$$\lambda\Phi(z) + (1 - \lambda)\Phi(z') > \Phi(\lambda z + (1 - \lambda)z')$$

for all $z, z' \in \mathbb{R}^N$ and $\lambda \in (0, 1)$. A typical example of symmetric strictly convex functions is the square-sum $\Phi(z) = \sum_{i \in N} z_i^2$.

In general, for $z \in \mathbb{R}^N$ and $i, j \in N$ with $z_i > z_j$, we have $\Phi(z - \varepsilon(\chi_i - \chi_j)) < \Phi(z)$ for any $\varepsilon \in (0, z_i - z_j)$ because

$$\begin{aligned}\Phi(z) &= \lambda\Phi(z - (z_i - z_j)(\chi_i - \chi_j)) + (1 - \lambda)\Phi(z) \\ &> \Phi(\lambda(z - (z_i - z_j)(\chi_i - \chi_j)) + (1 - \lambda)z) = \Phi(z - \lambda(z_i - z_j)(\chi_i - \chi_j)) \qquad (2)\end{aligned}$$

for any $\lambda \in (0, 1)$. Here, $\chi_i$ represents a unit vector where only the $i$th component is equal to 1, while all other components are equal to 0.

### 2.1 Relationship to fair allocation

Let $N = [n]$ represent the set of $n$ agents. We have two types of goods: $M = \{g_1, g_2, \dots, g_m\}$ represents the set of indivisible goods, and $C = \{c_1, c_2, \dots, c_r\}$ denotes the set of homogeneous divisible goods, that is, the valuation for a piece of a good is proportional to its fraction. The set of all goods is denoted by $E = M \cup C$. Let $v_{ie}$ be the valuation of good $e \in E$ for agent $i \in N$. We assume that agents have binary valuations, that is, the valuation $v_{ie}$ for the whole of good $e$ is either 0 or 1 for all $i \in N$ and $e \in E$. An instance of the fair allocation we deal with in this paper is described as $(N, M, C, v)$. Without loss of generality, we assume that, for any $e \in E$, there exists $i \in N$ such that $v_{ie} = 1$.

A *relaxed allocation* is defined as a matrix $\pi \in [0,1]^{N \times E}$ that satisfies (i) $\sum_{i \in N} \pi_{ie} = 1$ for all $e \in E$ and (ii) $\pi_{ie} = 0$ for any $i \in N$ and $e \in E$ with $v_{ie} = 0$. In a relaxed allocation $\pi$, each agent $i$ receives each good $e$ in the proportion of $\pi_{ie}$. Relaxed allocations treat indivisible goods as divisible. A relaxed allocation $\pi$ is an *allocation* if it additionally satisfies $\pi_{ie} \in \{0,1\}$ for all $i \in N$ and $e \in M$. A relaxed allocation $\pi$ is an *integral allocation* if it additionally satisfies $\pi_{ie} \in \{0,1\}$ for all $i \in N$ and $e \in E$. For an allocation $\pi$, an agent $i \in N$, and a subset of goods $E' \subseteq E$, let $\pi_i(E') = \sum_{e \in E'} \pi_{ie}$, which is the valuation of agent $i$'s bundle from $E'$. For an allocation $\pi$, the utility of agent $i \in N$ is defined as $\pi_i(E)$. For an allocation $\pi$, let $\pi(E)$ be the utility vector $(\pi_1(E), \ldots, \pi_n(E))$.

Here we rewrite the set of possible utility vectors in terms of an integral base-polyhedron. We define $f_M, f_C, f_E \colon 2^N \to \mathbb{Z}_+$ as follows: for a subset $X \subseteq N$ of agents,

- $f_M(X) = |\{g \in M : v_{ig} = 0\ (\forall i \notin X)\}|$ is the number of indivisible goods that must be allocated to agents in $X$,
- $f_C(X) = |\{c \in C : v_{ic} = 0\ (\forall i \notin X)\}|$ is the number of divisible goods that must be allocated to agents in $X$, and
- $f_E(X) = f_M(X) + f_C(X)$ is the number of goods that must be allocated to agents in $X$.

It is not difficult to see that the functions $f_M, f_C, f_E$ are normalized integer-valued supermodular.[2] Let $\ddot{\mathbf{B}}_M$ and $\overline{\mathbf{B}}_C$ be the M-convex set of $f_M$ and the integral base-polyhedron of $f_C$, respectively. In addition, let $\mathbf{B}_E$ be the Minkowski sum of $\ddot{\mathbf{B}}_M$ and $\overline{\mathbf{B}}_C$, i.e., $\mathbf{B}_E = \{x + y : x \in \ddot{\mathbf{B}}_M,\ y \in \overline{B}_C\}$. Then, $\mathbf{B}_E$ is the set of possible utility vectors.

Recall that $\overline{\mathbf{B}}_M = \mathrm{conv}(\ddot{\mathbf{B}}_M)$ and $\ddot{\mathbf{B}}_C = \overline{B}_C \cap \mathbb{Z}^N$. We denote $\overline{\mathbf{B}}_E = \mathrm{conv}(\mathbf{B}_E)$, and $\ddot{\mathbf{B}}_E = \mathbf{B}_E \cap \mathbb{Z}^N$. Note that $\mathbf{B}_E$ is not necessarily an M-convex set or an integral base-polyhedron as we have seen in Example 1.

For a symmetric strictly convex function $\Phi$, an allocation $\pi$ is called $\Phi$-*fair* if the utility vector $(\pi_1(E), \ldots, \pi_n(E))$ minimizes $\Phi$ among allocations.

Some prominent fairness notions are naturally represented as $\Phi$-fairness for some $\Phi$. An allocation $\pi$ is said to achieve the *maximum Nash welfare* (MNW) if the number of agents with positive utilities is maximized, and subject to that, the Nash welfare $(\prod_{i \in N : \pi_i(E) > 0} \pi_i(E))^{1/n}$ is maximized. Finding a utility vector of an MNW allocation is equivalent to minimizing $-\prod_{i \in N : z_i > 0}(z_i + \varepsilon)$ for some sufficiently small $\varepsilon > 0$ (see Appendix A in the full version [29]). The egalitarian social welfare is defined by the smallest utility among agents. Maximizing the egalitarian social welfare is a weaker notion of *increasingly maximal* (*inc-max*, for short) allocations; an allocation is inc-max if its smallest utility is as large as possible, within this, its second smallest utility is as large as possible, and so on. Similarly, we say that an allocation is *decreasingly minimal* (*dec-min*, for short) if its largest utility is as small as possible, within this, its second largest utility is as small as possible, and so on. We show that a certain symmetric strictly convex function $\Phi$ induces the dec-min and inc-max solution as a $\Phi$-fair allocation.

[2] Most of our results can be extended to the case when each agent evaluates indivisible goods with a matroid rank function and divisible goods with the concave closure of a matroid rank function because the functions $f_M, f_C, f_E$ continue to be normalized integer-valued supermodular in this scenario.

▶ **Proposition 5.** *Let $(N, M, C, v)$ be a fair allocation instance. There exists a symmetric strictly convex function $\Phi$ such that a dec-min allocation is $\Phi$-fair. In addition, there exists a symmetric strictly convex function $\Phi'$ such that an inc-max allocation is $\Phi'$-fair.*

Then for a symmetric strictly convex function $\Phi\colon \mathbb{R}^N \to \mathbb{R}$, we can rewrite finding a utility vector $z^*$ of a $\Phi$-fair allocation as the hybrid problem (1). If $M = \emptyset$ or $C = \emptyset$, the problem (1) can be solved in polynomial time [15, 16, 17, 18, 34].

When there are only divisible goods or only indivisible goods, once we obtain a $\Phi$-mimizer $z^*$ of (1), a $\Phi$-fair allocation $\pi$ is obtained by using the maximum flow problem. Since we can find an integral maximum flow in the indivisible goods case, $\pi$ can be an integral allocation. However, when both types of goods exist, it is not straightforward to construct an allocation from a given utility vector. Indeed, given a vector $u$ (not necessarily in $\mathbf{B}_E$), checking the existence of an allocation whose utility vector is $u$ is NP-hard (see Appendix B in the full version [29]).

As mentioned in Introduction, when $\mathbf{B}_E = \overline{\mathbf{B}}_C$ (continuous case) or $\mathbf{B}_E = \ddot{\mathbf{B}}_M$ (discrete case), the set of $\Phi$-minimizers is independent of $\Phi$. However, this is not the case in general even in fair allocation with both types of goods (i.e., $M \neq \emptyset$ and $C \neq \emptyset$). Thus, our problem is challenging.

▶ **Example 6.** Consider an instance with five agents $N = \{1, 2, 3, 4, 5\}$, five indivisible goods $M = \{g_1, g_2, g_3, g_4, g_5\}$, and three divisible goods $C = \{c_1, c_2, c_3\}$. Suppose that agents 1, 2, 3, and 4 desire all the goods, but agent 5 desires only the indivisible goods (see Table 1). Then, an allocation $\pi$ with $\pi(E) = (7/4, 7/4, 7/4, 7/4, 1)$ is dec-min. However, an allocation $\rho$ with $\rho(E) = (6/4, 6/4, 6/4, 6/4, 2)$ is inc-max and square-sum minimizer. Indeed, $\sum_{i \in N} \pi_i(E)^2 = 13.25$ and $\sum_{i \in N} \rho_i(E)^2 = 13$.

▶ **Example 7.** Consider an instance with five agents $N = \{1, 2, 3, 4, 5\}$, five indivisible goods $M = \{g_1, g_2, g_3, g_4, g_5\}$, and *two* divisible goods $C = \{c_1, c_2\}$. Suppose that agents 1, 2, 3, and 4 desire all the goods, but agent 5 desires only the indivisible goods. Then, an allocation $\pi$ with $\pi(E) = (\frac{6}{4}, \frac{6}{4}, \frac{6}{4}, \frac{6}{4}, 1)$ is dec-min and square-sum minimizer. However, an allocation $\rho$ with $\rho(E) = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, 2)$ is inc-max. Indeed, $\sum_{i \in N} \pi_i(E)^2 = 10$ and $\sum_{i \in N} \rho_i(E)^2 = 10.25$.

**Table 1** Valuations in Example 6.

| agents | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

**Table 2** Valuations in Example 7.

| agents | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $c_1$ | $c_2$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |

## 2.2 Principal Partition and Canonical Partition

Consider the integral base-polyhedron $\overline{\mathbf{B}}$ and the M-convex set $\ddot{\mathbf{B}}$ of a supermodular function $f\colon 2^N \to \mathbb{Z}_+$. For any real number $\lambda$, let $\mathcal{L}(\lambda)$ be the set of all maximizers of $f(X) - \lambda|X|$, i.e., $\mathcal{L}(\lambda) = \arg\max_{X \subseteq N}(f(X) - \lambda|X|)$. Note that $\mathcal{L}$ has a lattice structure, i.e., $\mathcal{L}$ is closed under union and intersection. Let $\mathrm{L}(\lambda)$ be the smallest member in $\mathcal{L}(\lambda)$. It is known that $\mathrm{L}(\lambda) \subseteq \mathrm{L}(\lambda')$ for any $\lambda > \lambda'$ (see, e.g., [17, Proposition 3.1]).

Fujishige [18] characterized the optimal utility vectors by the *principal partition* of $N$. There are at most $|N|$ number of $\lambda$ for which $|\mathcal{L}(\lambda)| \geq 2$. Let us denote such numbers as $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, which are called the *critical values*. The principal partition $\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_r$ is a partition of $N$ defined by

$$\hat{N}_j = \mathrm{L}(\lambda'_j) - \mathrm{L}(\lambda_j) \quad (j = 1, 2, \ldots, r),$$

where $\lambda'_j$ is an arbitrary real satisfying $\lambda_j > \lambda'_j > \lambda_{j+1}$ (assuming that $\lambda_{r+1} = -\infty$).

▶ **Theorem 8** (Fujishige [18] and Maruyama [34]). *The unique minimizer $x^*$ of $\min_{x \in \overline{\mathbf{B}}} \Phi(x)$ satisfies $x^*_i = \lambda_j$ for each $i \in \hat{N}_j$ and $j \in [r]$.*

The principal partition and critical values can be found in strongly polynomial time by using the submodular function minimization [27,42]; see also [38]. For more details of the principal partition, see a book and a survey of Fujishige [19,20].

Frank and Murota [15] characterized the optimal utility vectors of $\min_{y \in \ddot{\mathbf{B}}} \Phi(y)$ by the *canonical partition* of $N$. There are at most $|N|$ number of $\beta \in \mathbb{Z}$ for which $\mathrm{L}(\beta) \neq \mathrm{L}(\beta - 1)$. Let us denote such numbers as $\beta_1 > \beta_2 > \cdots > \beta_q$, which are called the *essential values*. The canonical partition $N_1, N_2, \ldots, N_q$ is a partition of $N$ defined by

$$N_i = \mathrm{L}(\beta_i - 1) - \mathrm{L}(\beta_i) \quad (i = 1, 2, \ldots, q).$$

Alternatively, the canonical partition and the essential values can be obtained by the following procedure [17, Section 3]: for $j = 1, 2, \ldots, q$, define

$$\beta_j = \max_{\emptyset \neq X \subseteq N \setminus \bigcup_{j'=1}^{j-1} N_{j'}} \left\lceil (f(X \cup \textstyle\bigcup_{j'=1}^{j-1} N_{j'}) - f(\bigcup_{j'=1}^{j-1} N_{j'}))/|X| \right\rceil, \tag{3}$$

$$h_j(X) = f(X \cup \textstyle\bigcup_{j'=1}^{j-1} N_{j'}) - (\beta_j - 1)|X| - f(\bigcup_{j'=1}^{j-1} N_{j'}) \quad (\forall X \subseteq N \setminus \bigcup_{j'=1}^{j-1} N_{j'}),$$

$$N_j = \text{smallest subset of } N \setminus \textstyle\bigcup_{j'=1}^{j-1} N_{j'} \text{ maximizing } h_j.$$

They provided a strongly polynomial-time algorithm to compute the canonical partition and the essential values by using this structure and a strongly polynomial-time algorithm for the submodular function minimization [27,42].

▶ **Theorem 9** (Frank and Murota [15,16,17]). *The essential values $\beta_1 > \beta_2 > \cdots > \beta_q$ are obtained from the critical values $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ as the distinct members of the rounded-up integers $\lceil \lambda_1 \rceil \geq \lceil \lambda_2 \rceil \geq \cdots \geq \lceil \lambda_r \rceil$. Moreover, the canonical partition is an aggregation of the principal partition as $N_i = \bigcup_{j \colon \lceil \lambda_j \rceil = \beta_i} \hat{N}_j$ for each $i \in [q]$. Any minimizer $y^*$ of $\min_{y \in \ddot{\mathbf{B}}} \Phi(y)$ satisfies $\beta_j - 1 \leq y^*_i \leq \beta_j$ for each $i \in N_j$ and $j \in [q]$. The minimizer $y^*$, the canonical partition, and essential values can be found in strongly polynomial time with respect to $|N|$.*

## 3 Exchange properties of integral base-polyhedra and M-convex sets

In this section, we review exchange properties of an integral base-polyhedron and an M-convex set. We also add new exchange properties for our hybrid case.

Let $f \colon 2^N \to \mathbb{Z}$ be a supermodular function. Let $\overline{\mathbf{B}}$ be an integral base-polyhedra and let $\ddot{\mathbf{B}}$ be an M-convex set defined by $f$. It is known that the M-convex set $\ddot{\mathbf{B}}$ and the integral base-polyhedron $\overline{\mathbf{B}}$ satisfy the *exchange properties*, respectively. For a vector $z \in \mathbb{R}^N$, define $\mathrm{supp}^+(z) := \{i \in N : z_i > 0\}$ and $\mathrm{supp}^-(z) := \{i \in N : z_i < 0\}$.

▶ **Proposition 10** ([37]). *For any $x, y \in \ddot{\mathbf{B}}$ and $i \in \mathrm{supp}^+(x - y)$, there exists some $j \in \mathrm{supp}^-(x - y)$ such that $x - \chi_i + \chi_j \in \ddot{\mathbf{B}}$ and $y + \chi_i - \chi_j \in \ddot{\mathbf{B}}$.*

▶ **Proposition 11** ([37]). *For any $x, y \in \overline{\mathbf{B}}$ and $i \in \mathrm{supp}^+(x - y)$, there exists some $j \in \mathrm{supp}^-(x - y)$ and a positive real $\alpha_0$ such that $x - \alpha(\chi_i - \chi_j) \in \overline{\mathbf{B}}$ and $y + \alpha(\chi_i - \chi_j) \in \overline{\mathbf{B}}$ for all $\alpha \in [0, \alpha_0]$.*

In addition, we show the following variants of exchange properties.

▶ **Proposition 12.** *For any $x, y \in \ddot{\mathbf{B}}$ and $X \subseteq N$ with $x(X) > y(X)$, there exist $i \in X$ and $j \in N \setminus X$ such that $x - \chi_i + \chi_j \in \ddot{\mathbf{B}}$. Moreover, if $x(X) > f(X)$, there exist $i \in X$ and $j \in N \setminus X$ such that $x - \chi_i + \chi_j \in \ddot{\mathbf{B}}$.*

▶ **Proposition 13.** *For any $x, y \in \overline{\mathbf{B}}$ and $X \subseteq N$ with $x(X) > y(X)$, there exist $i \in X$, $j \in N \setminus X$, and $\varepsilon > 0$ such that $x - \varepsilon(\chi_i - \chi_j) \in \overline{\mathbf{B}}$. Moreover, if $x(X) > f(X)$, there exist $i \in X$, $j \in N \setminus X$, and $\varepsilon > 0$ such that $x - \varepsilon(\chi_i - \chi_j) \in \overline{\mathbf{B}}$.*

▶ **Proposition 14.** *For any $x \in \ddot{\mathbf{B}}$ and disjoint subsets $I, J \subsetneq N$ such that $x(J) < f(I \cup J) - f(I)$, there exist $i \in I$ and $j \in J$ such that $x - \chi_i + \chi_j \in \ddot{\mathbf{B}}$.*

Then we show an exchange property for the hybrid situation.

▶ **Proposition 15.** *For any $x \in \ddot{\mathbf{B}}$ and $y \in \overline{\mathbf{B}}$, and $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x-y)$ such that $x-\chi_i+\chi_j \in \ddot{\mathbf{B}}$. Also, for any $x \in \ddot{\mathbf{B}}$ and $y \in \overline{\mathbf{B}}$, and $i \in \mathrm{supp}^-(x-y)$, there exists $j \in \mathrm{supp}^+(x - y)$ such that $x + \chi_i - \chi_j \in \ddot{\mathbf{B}}$.*

**Proof.** We only provide a proof for the former part, as the latter part can be demonstrated in a similar manner. By Proposition 11, there exists some $j \in \mathrm{supp}^-(x - y)$ and a positive real $\alpha_0$ such that $x' := x - \alpha_0(\chi_i - \chi_j) \in \overline{\mathbf{B}}$. For any $X \subseteq N$ with $i \in X$ and $j \notin X$, we have $x'(X) = x(X) - \alpha_0 \geq f(X)$, and hence $x(X) - 1 \geq f(X)$ since $x(X)$ and $f(X)$ are integers. Therefore, $x - \chi_i + \chi_j \in \ddot{B}$ holds. ◀

## 4 Structure of Φ-minimizers

In this section, we prove a proximity theorem (Theorem 2) by using the structure (Lemma 28) based on the canonical partition. Moreover, in the case of fair allocation, we also show that the problem has a canonical partition of goods (Theorem 29).

Let $f_M$ and $f_C$ be two supermodular functions over $M$ and $C$, respectively. Let also $f_E = f_M + f_C$. Recall that $\ddot{\mathbf{B}}_M$ and $\overline{\mathbf{B}}_C$ are a corresponding M-convex set and integral base-polyhedron, respectively. In addition, $\mathbf{B}_E$ is the Minkowski sum of $\ddot{\mathbf{B}}_M$ and $\overline{\mathbf{B}}_C$.

It should be noted that, for the integral base-polyhedron $\overline{\mathbf{B}}$ and the M-convex set $\ddot{\mathbf{B}}$ of a common supermodular function, the following proximity theorem has been shown by Frank and Murota [17].

▶ **Theorem 16** ([17, Theorem 4.1]). *Let $\Phi$ be a symmetric strictly convex function. For any $x^* \in \arg\min_{x \in \ddot{\mathbf{B}}} \Phi(x)$ and $y^* \in \arg\min_{y \in \overline{\mathbf{B}}} \Phi(y)$, we have $\lfloor y_i^* \rfloor \leq x_i^* \leq \lceil y_i^* \rceil$ for all $i \in N$.*

Note that this is a special case of our Theorem 2 when $f_C(X) = 0$ $(\forall X \subseteq N)$. We prove Theorem 2 following the same approach as for Theorem 16. However, we need to conduct a more detailed analysis to handle $\overline{\mathbf{B}}_C$.

Throughout this section, we fix a symmetric strictly convex function $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ and its minimizer $z^* \in \arg\min_{z \in \mathbf{B}_E} \Phi(z)$. By definition, $z^*$ can be represented as $x^* + y^*$ by $x^* \in \ddot{\mathbf{B}}_M$ and $y^* \in \overline{\mathbf{B}}_C$. Moreover, let $N_1, \ldots, N_q$ and $\beta_1, \ldots, \beta_q$ be the canonical partition and the essential values of the M-convex set $\ddot{\mathbf{B}}_E$.

In subsequent subsections, we prove Theorem 2 through the following steps. First, in Section 4.1, we demonstrate that $z_i^*$ lies within the interval $[\beta_1 - 1, \beta_1]$ for $i \in N_1$. Then, in Section 4.2, we decompose the problem of finding $z^*$ into two independent problems on $N_1$ and $N \setminus N_1$. By iteratively applying the same procedure, we obtain the desired structure. In Section 4.3, we show an additional result when $\mathbf{B}_E$ emerges from the fair allocation.

## 4.1 The peak-set $N_1$

In this subsection, we prove that $\beta_1 - 1 \le z_i^* \le \beta_1$ ($\forall i \in N_1$) and $z^*(N_1) = f_E(N_1)$. The main idea to prove these is to transfer some amounts from elements with high value to those with low value, which improves the objective value. Here, we need to be careful about the constraint.

We first introduce the basic properties for an M-convex set $\ddot{\mathbf{B}}$ and an integral base-polyhedron $\overline{\mathbf{B}}$.

▶ **Proposition 17.** *For any $x \in \ddot{\mathbf{B}}$, if $x - \chi_i + \chi_j \in \ddot{\mathbf{B}}$ and $x - \chi_j + \chi_k \in \ddot{\mathbf{B}}$, then it holds that $x - \chi_i + \chi_k \in \ddot{\mathbf{B}}$.*

▶ **Proposition 18** ([19, Lemma 4.5])**.** *Let $x \in \ddot{\mathbf{B}}$. Suppose that there exist a sequence $i_1, j_1, \dots, i_r, j_r$ of $2r$ distinct elements in $N$ such that $x - \chi_{i_h} + \chi_{j_k} \in \ddot{\mathbf{B}}$ if $h = k$ and $x - \chi_{i_h} + \chi_{j_k} \notin \ddot{\mathbf{B}}$ if $h > k$ for $h, k \in [r]$. Then, it holds that $x - \sum_{k \in [r]} (\chi_{i_k} - \chi_{j_k}) \in \ddot{\mathbf{B}}$.*

▶ **Proposition 19** ([37])**.** *For any $x \in \overline{\mathbf{B}}$, we have $x \in \mathrm{conv}(\overline{\mathbf{B}} \cap \{y \in \mathbb{Z}^N : \|x - y\|_\infty < 1\})$.*

▶ **Lemma 20.** *If $y^* + \varepsilon(\chi_i - \chi_j) \in \overline{\mathbf{B}}_C$ for some $i, j \in N$ and $\varepsilon > 0$, then $z_i^* = x_i^* + y_i^* \ge x_j^* + y_j^* = z_j^*$. In addition, for any $\beta \in \mathbb{R}$, it holds that $y^*(N') = f_C(N')$ with $N' = \{i \in N : z_i^* \ge \beta\}$.*

**Proof.** For the former statement, suppose to the contrary that $y^* + \varepsilon(\chi_i - \chi_j) \in \overline{\mathbf{B}}_C$ for some $i, j \in N$ such that $x_i^* + y_i^* < x_j^* + y_j^*$ and $\varepsilon > 0$. Then, $y = y^* + \min\{\varepsilon, (y_j^* - y_i^*)/2\} \cdot (\chi_i - \chi_j) \in \overline{\mathbf{B}}_C$ and $\Phi(x^* + y) < \Phi(x^* + y^*)$ by (2). This contradicts the assumption that $z^* = x^* + y^*$ is a $\Phi$-minimizer.

For the latter statement, suppose that $y^*(N') > f_C(N')$ for some $N' = \{i \in N : z_i^* \ge \beta\}$ with $\beta \in \mathbb{R}$. Then, by Proposition 13, there exist $i \in N'$, $j \in N \setminus N'$, and $\varepsilon > 0$ such that $y^* + \varepsilon(\chi_j - \chi_i) \in \overline{\mathbf{B}}_C$. By the former statement, this implies $z_j^* \ge z_i^*$, contradicting $j \notin N'$. ◀

We define a graph whose edge represents that transferring a unit amount of elements does not violate the constraints. Then we show that transferring a unit amount along a path will improve the objective value.

▶ **Lemma 21.** *Let $\beta$ be an integer and $N' \subseteq N$. Define $N^> = \{i \in N : z_i^* > \beta\}$, $N^= = \{i \in N : z_i^* = \beta\}$, and $N^< = \{i \in N : z_i^* < \beta\}$. Construct a graph*

$$G = \left(N', \left\{(i,j) \in N' \times N' : x^* - \chi_i + \chi_j \in \ddot{\mathbf{B}}_M \text{ or } y^* - \chi_i + \chi_j \in \overline{\mathbf{B}}_C\right\}\right).$$

*If $G$ has a path from some $i \in N' \cap N^>$ to some $j \in N' \cap N^>$ with $i \ne j$, then there exists a vector $z''$ such that $\Phi(z'') < \Phi(z^*)$.*

**Proof.** We first observe that Lemma 20 implies $y^*(N^>) = f_C(N^>)$ and $y^*(N^> \cup N^=) = f_C(N^> \cup N^=)$.

Let $P = (i_1, i_2, \dots, i_k)$ be a shortest path from some $i_1 \in N' \cap N^>$ to some $i_k \in N' \cap N^<$. Then we have $i_1 \in N^>$, $i_2, \dots, i_{k-1} \in N^=$, and $i_k \in N^<$. Since $z_{i_2}^* < z_{i_1}^*$, Lemma 20 implies that $y^* - \chi_{i_1} + \chi_{i_2} \notin \overline{\mathbf{B}}_C$, and thus $x^* - \chi_{i_1} + \chi_{i_2} \in \ddot{\mathbf{B}}_M$. If $x^* - \chi_{i_2} + \chi_{i_3} \in \ddot{\mathbf{B}}_M$, then $x^* - \chi_{i_1} + \chi_{i_3} \in \ddot{\mathbf{B}}_M$ by Proposition 17, which leads to a shortcut of $P$. Thus, we have $x^* - \chi_{i_2} + \chi_{i_3} \notin \ddot{\mathbf{B}}_M$ and instead, $y^* - \chi_{i_2} + \chi_{i_3} \in \overline{\mathbf{B}}_C$ holds. Next, let us assume that $y^* - \chi_{i_3} + \chi_{i_4} \in \overline{\mathbf{B}}_C$ and $i_4 \in N^=$, and derive a contradiction. Consider

$$\overline{\mathbf{B}}^= = \left\{ \tilde{y} \in \mathbb{R}^{N^=} : \begin{array}{l} \tilde{y}(N^=) = f_C(N^= \cup N^>) - f_C(N^>), \\ \tilde{y}(X) \ge f_C(X \cup N^>) - f_C(N^>) \ (\forall X \subseteq N^=) \end{array} \right\},$$

which is also an integral base-polyhedron of a supermodular function. Since $\beta$ is an integer, $y^*_{N^=}$ is also integral. Thus, the vectors $y^*_{N^=} - \chi_{i_2} + \chi_{i_3}$ and $y^*_{N^=} - \chi_{i_3} + \chi_{i_4}$ are contained in $\ddot{\mathbf{B}}^=$, since $y^*(N^>) = f_C(N^>)$ and $i_2, i_3, i_4 \in N^=$. Then $y^*_{N^=} - \chi_{i_2} + \chi_{i_4} \in \ddot{\mathbf{B}}^=$ follows from Proposition 17, which means that $y^* - \chi_{i_2} + \chi_{i_4} \in \overline{\mathbf{B}}_C$[3]. However, this implies a shortcut of $P$. Therefore, if $i_4 \in N^=$, then $y^* - \chi_{i_3} + \chi_{i_4} \notin \overline{\mathbf{B}}_C$, and hence $x^* - \chi_{i_3} + \chi_{i_4} \in \ddot{\mathbf{B}}_M$. By the same argument, we have $x^* - \chi_{i_\ell} + \chi_{i_{\ell+1}} \in \ddot{\mathbf{B}}_M$ if $\ell$ is an odd number and $i_{\ell+1} \in N^=$, and $y^* - \chi_{i_\ell} + \chi_{i_{\ell+1}} \in \overline{\mathbf{B}}_C$ if $\ell$ is an even number. Note that $k$ must be an even number, because $y^* - \chi_{i_{k-1}} + \chi_{i_k} \notin \overline{\mathbf{B}}_C$ follows from Lemma 20 and $z^*_{i_k} < z^*_{i_{k-1}}$. Moreover, for any integers $\ell$ and $h$ with $\ell \geq h + 2$, we have $x^* - \chi_{i_\ell} + \chi_{i_h} \notin \ddot{\mathbf{B}}_M$ and $y^* - \chi_{i_\ell} + \chi_{i_h} \notin \overline{\mathbf{B}}_C$. Let $x' = x^* - \sum_{\ell\colon \mathrm{odd}}(\chi_{i_\ell} - \chi_{i_{\ell+1}})$, $y' = y^* - \sum_{\ell\colon \mathrm{even}}(\chi_{i_\ell} - \chi_{i_{\ell+1}})$, and $z' = x' + y'$. By Proposition 18, we have $x' \in \ddot{\mathbf{B}}_M$, $y' \in \overline{\mathbf{B}}_C$, and $z' \in \mathbf{B}_E$. Note that $z' = z^* - \chi_{i_1} + \chi_{i_k}$, $y'(N^>) = y^*(N^>)$ $(= f_C(N^>))$ and $y'(N^> \cup N^=) = y^*(N^> \cup N^=)$ $(= f_C(N^> \cup N^=))$ by the construction. For notational convenience, we denote $i^* = i_1$ and $j^* = i_k$ in the following.

If $z^*_{i^*} > z^*_{j^*} + 1$, then $\Phi(z') < \Phi(z^*)$. Thus, suppose that $z^*_{i^*} \leq z^*_{j^*} + 1$ $(< \beta + 1)$. In this case, $\beta - 1 < z'_{i^*} < \beta$ and $\beta < z'_{j^*} < \beta + 1$. By Proposition 19, $y'$ can be represented by a convex combination of its integral neighbors in $\overline{\mathbf{B}}_C$. Let $y' = \sum_{t=1}^r \lambda^{(t)} \cdot y^{(t)}$, where $y^{(t)} \in \ddot{\mathbf{B}}_C \cap \{y \in \mathbb{Z}^N : \|y - y'\| < 1\}$ $(\forall t \in [r])$, $\sum_{t=1}^r \lambda^{(t)} = 1$, and $\lambda^{(t)} \geq 0$ $(\forall t \in [r])$. Define $z^{(t)} = x' + y^{(t)}$ for each $t$. Thus, we also obtain $z' = \sum_{t=1}^r \lambda^{(t)} \cdot z^{(t)}$. Note that $z^{(t)}_{i^*} \in \{\beta - 1, \beta\}$ and $z^{(t)}_{j^*} \in \{\beta, \beta + 1\}$ for each $t$. In addition, for each $t$, it holds that $y^{(t)}(N^>) = f_C(N^>)$ because $\sum_{t=1}^r \lambda^{(t)} \cdot y^{(t)}(N^>) = y'(N^>) = y^*(N^>) = f_C(N^>)$ and $y^{(t)}(N^>) \geq f_C(N^>)$. Similarly, we can see that $y^{(t)}(N^> \cup N^=) = f_C(N^> \cup N^=)$ for each $t$.

Let us choose an arbitrary $t$ with $z^{(t)}_{i^*} = \beta - 1$. Let

$$\overline{\mathbf{B}}^> = \{\tilde{y} \in \mathbb{R}^{N^>} : \tilde{y}(N^>) = f_C(N^>) \text{ and } \tilde{y}(X) \geq f_C(X)\ (\forall X \subseteq N^>)\}$$

(the restriction of $\overline{\mathbf{B}}_C$ to $N^>$), and $\ddot{\mathbf{B}}^>$ be the M-convex set induced from $\overline{\mathbf{B}}^>$. Then it holds that $y^{(t)}_{N^>} \in \ddot{\mathbf{B}}^>$, $y'_{N^>} \in \overline{\mathbf{B}}^>$ and $i^* \in \mathrm{supp}^-(y^{(t)}_{N^>} - y'_{N^>})$. We apply Proposition 15 to them. Then we can choose an index $i^{(t)} \in \mathrm{supp}^+(y^{(t)}_{N^>} - y'_{N^>})$ such that $y^{(t)}_{N^>} + \chi_{i^*} - \chi_{i^{(t)}} \in \overline{\mathbf{B}}^>$. We show that this implies $\hat{y}^{(t)} \coloneqq y^{(t)} + \chi_{i^*} - \chi_{i^{(t)}} \in \overline{\mathbf{B}}_C$. Indeed, for any $X$ with $i^* \notin X$ and $i^{(t)} \in X$ (the other cases are trivial), since $y^{(t)}(X \cap N^>) - 1 \geq f_C(X \cap N^>)$, we have

$$\begin{aligned} y^{(t)}(X) &= y^{(t)}(X \cap N^>) + y^{(t)}(X \cup N^>) - y^{(t)}(N^>) \\ &> f_C(X \cap N^>) + f_C(X \cup N^>) - f_C(N^>) \geq f_C(X), \end{aligned}$$

which implies that $\hat{y}^{(t)}(X) \geq f_C(X)$. In addition, we observe that $z^{(t)}_{i^{(t)}} = y^{(t)}_{i^{(t)}} + x'_{i^{(t)}} > y'_{i^{(t)}} + x'_{i^{(t)}} = z'_{i^{(t)}} = z_{i^{(t)}} > \beta$. Thus, since $z^{(t)}_{i^{(t)}}$ is an integer,

$$\hat{y}^{(t)}_{i^{(t)}} + x'_{i^{(t)}} = z^{(t)}_{i^{(t)}} - 1 \geq \beta. \tag{4}$$

On the other hand, for each $t$ with $z^{(t)}_{i^*} = \beta$, we denote $\hat{y}^{(t)} = y^{(t)}$. We also remark that $\hat{y}^{(t)}(N^> \cup N^=) = y^{(t)}(N^> \cup N^=)$ and $\hat{y}^{(t)}_i = y^{(t)}_i$ for each $t$ and $i \in N \setminus N^>$.

We will do similar operations for indices in $N^<$. Let us choose an arbitrary $t$ with $z^{(t)}_{j^*} = \beta_1 + 1$. We show that we can choose $j^{(t)} \in N^<$ with $z^{(t)}_{j^{(t)}} \leq \beta_1 - 1$ such that $\hat{y}^{(t)} - \chi_{j^*} + \chi_{j^{(t)}} \in \overline{\mathbf{B}}_C$ by applying Proposition 15. We denote $N^\geq = N^> \cup N^=$. Let also

$$\overline{\mathbf{B}}^< = \left\{ \tilde{y} \in \mathbb{R}^{N^<} : \begin{array}{l} \tilde{y}(N^<) = f_C(N) - f_C(N^\geq), \\ \tilde{y}(X) \geq f_C(X \cup N^\geq) - f_C(N^\geq)\ (\forall X \subseteq N^<) \end{array} \right\}.$$

[3] We need to check for each $X$ such that $i_2 \in X$ but $i_4 \notin X$. The case when $X \supseteq \{i_2, i_3\}$ follows by $y^* - \chi_{i_3} + \chi_{i_4} \in \overline{\mathbf{B}}_C$, and the case when $i_2 \in X$ but $i_3 \notin X$ follows by $y^* - \chi_{i_2} + \chi_{i_3} \in \overline{\mathbf{B}}_C$.

(the contraction of $\overline{\mathbf{B}}_C$ by $N^{\geq}$), and $\ddot{\mathbf{B}}^{<}$ be the M-convex set induced from $\overline{\mathbf{B}}^{<}$. Then it holds that $\hat{y}^{(t)}_{N^<} \in \ddot{\mathbf{B}}^{<}$, $y'_{N^<} \in \overline{\mathbf{B}}^{<}$ and $j^* \in \mathrm{supp}^+(\hat{y}^{(t)}_{N^<} - y'_{N^<})$. We apply Proposition 15 to them. Then we can choose an index $j^{(t)} \in \mathrm{supp}^-(\hat{y}^{(t)}_{N^<} - y'_{N^<})$ such that $\hat{y}^{(t)}_{N^<} - \chi_{j^*} + \chi_{j^{(t)}} \in \overline{\mathbf{B}}^{<}$. Then we can observe that $\hat{y}^{(t)} - \chi_{j^*} + \chi_{j^{(t)}} \in \overline{\mathbf{B}}_C$. Indeed, for any $X$ with $j^* \in X$ and $j^{(t)} \notin X$, since $y^{(t)}(X \cap N^<) - 1 \geq f_C((X \cap N^<) \cup N^{\geq}) - f_C(N^{\geq})$ and $y^{(t)}(X \cap N^{\geq}) \geq f_C(X \cap N^{\geq})$, we have

$$\begin{aligned}y^{(t)}(X) &= y^{(t)}(X \cap N^<) + y^{(t)}(X \cap N^{\geq}) \\ &> f_C(X \cap N^<) - f_C(N^{\geq}) + f_C(X \cup N^{\geq}) \geq f_C(X).\end{aligned}$$

Moreover, $z^{(t)}_{j^{(t)}} < y'_{j^{(t)}} + x'_{j^{(t)}} = z'_{j^{(t)}} = z_{j^{(t)}} < \beta$, which implies that

$$\hat{y}^{(t)}_{j^{(t)}} + 1 + x'_{j^{(t)}} = z^{(t)}_{j^{(t)}} + 1 \leq \beta. \tag{5}$$

For simplicity, let $j^{(t)} = j^*$ for each $t$ with $z^{(t)}_{j^*} = \beta$. Then, $y'' := \sum_{t=1}^{r} \lambda^{(t)} \cdot (\hat{y}^{(t)} - \chi_{j^*} + \chi_{j^{(t)}}) \in \overline{\mathbf{B}}_C$.

Let $z'' = x' + y''$. Note that this operation to produce $z''$ first reduces the value of elements more than $\beta$ while keeping them at least $\beta$ by (4), and then increases the value of elements less than $\beta$ while keeping them at most $\beta$ by (5). In other words, $\beta_1 \leq z''_i \leq z^*_i$ for $i \in N^>$, $z''_i = z^*_i$ for $i \in N^=$, and $z^*_i \leq z''_i \leq \beta_1$ for $i \in N^<$. Therefore, $\Phi(z'') < \Phi(z^*)$ holds. This contradicts to the optimality of $z^*$. ◀

To prove $\beta_1 - 1 \leq z^*_i \leq \beta_1$, we find a path on the graph by supposing the contrary. While it is easy for the continuous or discrete case, elaborate analysis is required in the hybrid case. Recall that $\beta_1 = \max\{\lceil f_E(X)/|X| \rceil : \emptyset \neq X \subseteq N\}$ by (3).

▶ **Lemma 22.** $z^*_i \leq \beta_1$ *for all* $i \in N$.

**Proof.** Define the sets $N^> = \{i \in N : z^*_i > \beta_1\}$, $N^= = \{i \in N : z^*_i = \beta_1\}$, and $N^< = \{i \in N : z^*_i < \beta_1\}$. By Lemma 20, $y^*(N^>) = f_C(N^>)$ and $y^*(N^> \cup N^=) = f_C(N^> \cup N^=)$.

Suppose to the contrary that $N^>$ is nonempty. We construct a graph

$$G = \big(N,\ \{(i,j) \in N^2 : x^* - \chi_i + \chi_j \in \ddot{\mathbf{B}}_M \text{ or } y^* - \chi_i + \chi_j \in \overline{\mathbf{B}}_C\}\big).$$

We observe that for any $Z$ with $N^> \subseteq Z \subseteq N^> \cup N^=$, it holds that

$$z^*(Z) > f_E(Z) \tag{6}$$

because $f_M(Z) + f_C(Z) = x^*(Z) + y^*(Z) = z^*(Z) > \beta_1 \cdot |Z| \geq f_E(Z) = f_M(Z) + f_C(Z)$, where the last inequality holds by the definition of $\beta_1$. This implies the following claim.

▷ Claim 23. For any $Z$ satisfying (6), there exists an edge $(i,j) \in Z \times (N \setminus Z)$.

▷ Claim 24. There exist paths in $G$ from some vertex in $N^>$ to some vertex in $N^<$.

By the above claim and Lemma 21 with $N' = N$ and $\beta = \beta_1$, there exists a vector $z$ with $\Phi(z) < \phi(z^*)$, which contradicts to the optimality of $z^*$. This completes the proof of Lemma 22. ◀

We then prove that $z^*_i$ is at least $\beta_1 - 1$ for all $i \in N_1$ by a similar technique to Lemma 22. Recall that $N_1$ is the smallest subset of $N$ maximizing $f_E(X) - (\beta_1 - 1)|X|$.

▶ **Lemma 25.** $z^*_i \geq \beta_1 - 1$ *for all* $i \in N_1$.

We then show that we cannot decrease the values of elements in $N_1$ (which have high values) anymore. In the words of fair allocation, this means that goods not required to be assigned to $N_1$ are not assigned to $N_1$.

▶ **Lemma 26.** *$x^*(N_1) = f_M(N_1)$, $y^*(N_1) = f_C(N_1)$, and $z^*(N_1) = f_E(N_1)$.*

**Proof.** It is sufficient to prove that $z^*(N_1) = f_E(N_1)$. Let $N^> := \{i \in N : z_i^* > \beta_1 - 1\}$, $N^= := \{i \in N : z_i^* = \beta_1 - 1\}$, and $N^< := \{i \in N : z_i^* < \beta_1 - 1\}$.

Suppose to the contrary that $z^*(N_1) > f_E(N_1)$. We construct a graph

$$G = \big(N, \big\{(i,j) \in N \times N : x^* - \chi_i + \chi_j \in \ddot{\mathbf{B}}_M \text{ or } y^* - \chi_i + \chi_j \in \overline{\mathbf{B}}_C\big\}\big).$$

Since $N_1 \in \arg\max_{S \subseteq N} f_E(S) - (\beta_1 - 1)|S|$, we have $f_E(N_1) - (\beta_1 - 1)|N_1| \geq f_E(X) - (\beta_1 - 1)|X|$ for any $X \subseteq N$. For any $X$ with $N^> \subseteq X \subseteq N^> \cup N^=$, it holds that

$$\begin{aligned}
z^*(X) &= z^*(N^>) + z^*(X \setminus N^>) = z^*(N^>) + (\beta_1 - 1) \cdot (|X| - |N^>|) \\
&\geq z^*(N^> \cap N_1) + (\beta_1 - 1) \cdot (|X| - |N^> \cap N_1|) \\
&= z^*(N_1) - (\beta_1 - 1) \cdot (|N_1| - |N^> \cap N_1|) + (\beta_1 - 1) \cdot (|X| - |N^> \cap N_1|) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 25)} \\
&= z^*(N_1) - (\beta_1 - 1)|N_1| + (\beta_1 - 1)|X| \\
&> f_E(N_1) - (\beta_1 - 1)|N_1| + (\beta_1 - 1)|X| \qquad \text{(by assumption)} \\
&\geq f_E(X) - (\beta_1 - 1)|X| + (\beta_1 - 1)|X| = f_E(X).
\end{aligned}$$

Hence, by the same proofs of Claims 23 and 24, there exist paths in $G$ from an agent in $N^>$ to an agent in $N^<$. Then, by applying Lemma 21 with $N' = N$ and $\beta = \beta_1 - 1$, we can decrease the value of $\Phi$, which is a contradiction. Hence, we obtain $z^*(N_1) = f_E(N_1)$. This implies that $x^*(N_1) = f_M(N_1)$ and $y^*(N_1) = f_C(N_1)$. ◀

## 4.2 Decomposition

We describe that we can derive a similar result for $N_2, \ldots, N_q$.

Let $N_1' = N \setminus N_1$. For a supermodular function $f$, we denote $f^{(1)} \colon 2^{N_1'} \to \mathbb{Z}$ to be the supermodular function obtained from $f$ by contracting $N_1$, i.e., $f^{(1)}(X) = f(X \cup N_1) - f(N_1)$ for each $X \subseteq N_1'$. We consider the M-convex set $\ddot{\mathbf{B}}_M^{(1)}$ of $f_M^{(1)}$, integral base-polyhedra $\overline{\mathbf{B}}_C^{(1)}$ of $f_C^{(1)}$, and $\mathbf{B}_E^{(1)} = \ddot{\mathbf{B}}_M^{(1)} + \overline{\mathbf{B}}_C^{(1)}$.

By Lemma 26, we have $z^*_{N_1'} \in \mathbf{B}_E^{(1)}$. In addition, for any $z_{N_1'} \in \mathbf{B}_E^{(1)}$, an extended vector $z = (z^*_{N_1}, z_{N_1'})$ is contained in $\mathbf{B}_E$ because $z(X) \geq f_E(X \cap N_1) + f_E^{(1)}(X \cap N_1') = f_E(X \cap N_1) + f_E(X \cup N_1) - f(N_1) \geq f_E(X)$ for all $X \subseteq N$ by the supermodularity of $f_E$. Hence, we obtain the following lemma. Let $\Phi' \colon \mathbb{R}^{N_1'} \to \mathbb{R}$ be the symmetric strictly convex function such that $\Phi'(z_{N_1'}) = \Phi(z^*_{N_1}, z_{N_1'})$.

▶ **Lemma 27.** *For any $z_{N_1'} \in \mathbf{B}_E^{(1)}$, a vector $z = (z^*_{N_1}, z_{N_1'})$ is a $\Phi$-minimizer of $\mathbf{B}_E$ if and only if $z_{N_1'}$ is a $\Phi'$-minimizer of $\mathbf{B}_E^{(1)}$.*

Therefore, we can apply the results in Section 4.1 to $N_1'$, $f_M^{(1)}$, $f_C^{(1)}$ and $f_E^{(1)}$, and repeat the same procedure. By the definition of the canonical partition and the essential values, we obtain the following lemma.

▶ **Lemma 28.** *For each $j = 1, \ldots, q$, it holds that $\beta_j - 1 \leq z_i^* \leq \beta_j$ for every $i \in N_j$.*

We can prove Theorem 2 by using Theorem 9 and Lemma 28, and basic results for the integral base-polyhedra.

**Proof of Theorem 2.** Recall that the partition $N_1, \dots, N_q$ is the canonical partition of $\ddot{\mathbf{B}}_E$, and $\beta_1, \dots, \beta_q$ are the corresponding essential values. Let $\hat{N}_1, \dots, \hat{N}_r$ be the principal partition of $\overline{\mathbf{B}}_E$, and let $\lambda_1, \dots, \lambda_r$ be the critical values. We fix $i \in N$ arbitrarily. Let $j \in [q]$ and $k \in [r]$ be the unique indices such that $i \in N_j$ and $i \in \hat{N}_k$. By invoking Theorem 9, we have $\beta_j = \lceil \lambda_k \rceil$. Consequently, by Theorem 8 and Lemma 28, we obtain $z^*_i \leq \beta_j = \lceil \lambda_k \rceil = \lceil \bar{z}_i \rceil$.

To show the other inequality, let $\Phi' \colon \mathbb{R}^N \to \mathbb{R}$ be a symmetric strictly convex function such that $\Phi'(z) = \Phi(-z)$ for $z \in \mathbb{R}^N$. Let $\mathbf{B}'_E = -\mathbf{B}_E$ and $\overline{\mathbf{B}}'_E = -\overline{\mathbf{B}}_E$. Then, $-z^*$ and $-\bar{z}$ are $\Phi'$-minimizers of $\mathbf{B}'_E$ and $\overline{\mathbf{B}}'_E$, respectively. By applying the same argument as above, we obtain $-z^*_i \leq \lceil -\bar{z}_i \rceil$, which is equivalent to $z^*_i \geq \lfloor \bar{z}_i \rfloor$. ◀

## 4.3 Structures in Fair Allocation

We establish the structure in the case of fair allocations. We partition the goods according to the canonical partition as follows. Let $M_1$ and $C_1$ denote the subset of indivisible goods $M$ and divisible goods $C$, respectively, that must be allocated to agents in $N_1$. We iteratively define $M_j$ and $C_j$ as the subset of $M \setminus \bigcup_{j'=1}^{j-1} M_{j'}$ and $C \setminus \bigcup_{j'=1}^{j-1} C_{j'}$, respectively, that must be allocated to agents in $\bigcup_{j'=1}^{j} N_{j'}$. In other words, $M_j$ and $C_j$ $(j = 1, \dots, q)$ is defined as

$$M_j = \left\{g \in M \setminus \textstyle\bigcup_{j'=1}^{j-1} M_{j'} : v_{ig} = 0 \ (\forall i \in N \setminus \bigcup_{j'=1}^{j} N_{j'})\right\}, \tag{7}$$

$$C_j = \left\{c \in C \setminus \textstyle\bigcup_{j'=1}^{j-1} C_{j'} : v_{ic} = 0 \ (\forall i \in N \setminus \bigcup_{j'=1}^{j} N_{j'})\right\}. \tag{8}$$

We refer $M_1, \dots, M_q$ and $C_1, \dots, C_q$ as the canonical partitions of the indivisible goods and the divisible goods, respectively. By Lemmas 26 and 27, we show the following.

▶ **Theorem 29.** *For any allocation $\pi^*$ whose utility vector is a $\Phi$-minimizer over $\mathbf{B}_E$, it holds that $\sum_{i \in N_j} \pi^*_{ie} = 1$ for every good $e \in M_j \cup C_j$ and $j = 1, 2, \dots, q$.*

**Proof.** Let $z^*$ be the utility vector of $\pi^*$. Let $x^*$ and $y^*$ be the utility vectors for indivisible and divisible goods in $\pi^*$, respectively. Thus, $z^* = x^* + y^*$.

First, since $f_M(N_1) = |M_1|$ and $f_C(N_1) = |C_1|$, we have $z^*(N_1) = |M_1 \cup C_1|$ by Lemma 26. Next, let $N'_j = N \setminus \bigcup_{j'=1}^{j} N_{j'}$ for $j = 1, \dots, q-1$. By Lemma 27, $z^*_{N'_1}$ is a $\Phi'$-minimizer, where $\Phi'(z) = \phi(z^*_{N_1}, z)$. Thus by the definition of $N_2$ and Lemma 26 again, $x^*_{N'_1}(N_2) = f^{(1)}_M(N_2) = f_M(N_1 \cup N_2) - f_M(N_1) = |M_2|$ and $y^*_{N'_1}(N_2) = f^{(1)}_C(N_2) = |C_2|$. By iteratively applying this argument, we observe that $z^*(N_j) = |M_j \cup C_j|$ for $j = 2, \dots, q$.

Because agents in $N_q$ want only the goods in $M_q \cup C_q$, these goods are allocated to the agents in $N_q$. Then, for each $j = q-1, \dots, 1$, since agents in $N_j$ want only the goods in $\bigcup_{j'=j}^{q} (M_{j'} \cup C_{j'})$ but the goods in $\bigcup_{j'=j+1}^{q} (M_{j'} \cup C_{j'})$ are allocated to agents in $N_{j+1} \cup \cdots \cup N_q$, the goods in $M_j \cup C_j$ are allocated to agents in $N_j$. Therefore, the theorem holds. ◀

# 5 Tractability for Identical Divisible Goods

In this section, we focus on the setting where all the divisible goods are identical, i.e., $v_{ic} = v_{ic'}$ for any $c, c' \in C$ and $i \in N$. Let $\Phi$ be a symmetric strictly convex function. The main result of this section is Theorem 4, i.e., a polynomial-time algorithm to find a $\Phi$-fair allocation. As a corollary, an MNW allocation for mixed goods can be found in polynomial time when the divisible goods are identical.

Our algorithm utilizes structures discussed in the previous section. Namely, we seek allocations whose utility vectors satisfy the statements in Lemma 28 and Theorem 29, which are necessary conditions to be optimal. The high-level idea of our algorithm is described as follows.

First, we find a discrete $\Phi$-minimizer $\ddot{z}^* \in \arg\min_{z \in \ddot{\mathbf{B}}_E} \Phi(z)$, the canonical partition $N_1, \ldots, N_q$, and the essential values $\beta_1, \ldots, \beta_q$ of $\ddot{\mathbf{B}}_E$. These can be found in polynomial time by Theorem 9. Note that $\ddot{z}^*$ is an optimal utility vector if every good is assumed to be indivisible. We also compute the canonical partition of the indivisible goods $M_1, \ldots, M_q$ and that of the divisible ones $C_1, \ldots, C_q$ by (7) and (8). Since all divisible goods are identical, only one of $C_1, \ldots, C_q$ can be non-empty. Let $j^*$ be the index such that $C_{j^*} = C$.

Next, we decide on an allocation for agents in $N_j$ for each $j \neq j^*$. Let us fix $j \neq j^*$. Theorem 29 implies that in an optimal allocation, the agents in $N_j$ receive only the indivisible goods in $M_j$. Moreover, by Lemma 28, some agents in $N_j$ must receive $\beta_j$ goods and the others must receive $\beta_j - 1$. Because $\beta_j - 1 \leq \ddot{z}_i^* \leq \beta_j$ holds by Theorem 9, we allocate goods in $M_j$ so that each agent $i \in N \setminus N_{j^*}$ receives $\ddot{z}_i^*$ goods. Such an allocation can be computed by solving a bipartite matching problem.[4] For agents in $N_j$, the utility vector of this allocation is the same as an optimal one up to arrangement of elements. Thus, we have found an optimal allocation for agents in $N_j$.

The remaining task is to determine the allocation of $M_{j^*} \cup C$ to agents $N_{j^*}$. Since the optimal allocation of $M_{j^*} \cup C$ depends on $\Phi$, we conduct an enumeration-based approach rather than performing a full characterization.

Let $\pi^*$ be an optimal allocation. Let $N_{j^*}^+$ be the set of agents in $N_{j^*}$ who desire the divisible goods, i.e., $N_{j^*}^+ = \{i \in N_{j^*} : v_i(c) = 1 \ (\forall c \in C)\}$. Let $N_{j^*}^- = N_{j^*} \setminus N_{j^*}^+$. The following lemma indicates that there are a finite number of candidates for a $\Phi$-minimizer.

▶ **Lemma 30.** *All the agents receiving divisible goods (i.e., $\pi_i^*(C) > 0$) have the same utility.*

Let $k$ be the number of agents in $N_{j^*}^+$ who receive $\beta_{j^*}$ indivisible goods and let $\ell$ be the total number of indivisible goods received by agents in $N_{j^*}^+$. Note that $k < |N_{j^*}^+|$. The key observation is the following lemma.

▶ **Lemma 31.** *The following properties hold:*

1. $|N_{j^*}^+| \cdot (\beta_{j^*} - 1) + k \leq \ell + |C| \leq |N_{j^*}^+| \cdot \beta_{j^*}$;
2. *there exist $X \subseteq N_{j^*}^+$ such that $|X| = k$ and*
   a. *for each $i \in X$: $\pi_i^*(M_{j^*}) = \beta_{j^*}$ and $\pi_i^*(C) = 0$;*
   b. *for each $i \in N_{j^*}^+ \setminus X$: $\pi_i^*(M_{j^*}) \leq \beta_{j^*} - 1$ and $\pi_i^*(E) = \beta_{j^*} - (|N_{j^*}^+| \cdot \beta_{j^*} - \ell - |C|)/(|N_{j^*}^+| - k)$;*
3. $\pi_i^*(M_{j^*}) \in \{\beta_{j^*}, \beta_{j^*} - 1\}$ *and* $\pi_i^*(C) = 0$ *for each* $i \in N_{j^*}^-$.

Since Lemma 31 specifies the utility vector (i.e., $\pi_i^*(E)$ for each $i \in N_{j^*}$) of an optimal allocation up to arrangement, it suffices to find an allocation whose utility vector satisfies the statement in Lemma 31. In fact, if we are given $\ell$, an optimal allocation can be computed as follows. For each $k = 0, \ldots, |N_{j^*}^+|$ such that property 1 in Lemma 31 is satisfied,

1. find an allocation $\pi^{k,\ell} \in \{0,1\}^{N_{j^*} \times M_{j^*}}$ of indivisible goods in $M_{j^*}$ such that (a) $|\{i \in N_{j^*}^+ : \pi_i(M_{j^*}) = \beta_{j^*}\}| \leq k$, (b) $\pi_i(M_{j^*}) \leq \beta_{j^*}$ for each $i \in N_{j^*}^+$, (c) $\sum_{i \in N_{j^*}^+} \pi_i(M_{j^*}) = \ell$, (d) $\pi_i(M_{j^*}) \in \{\beta_{j^*}, \beta_{j^*} - 1\}$ for each $i \in N_{j^*}^-$;

[4] It can also be calculated directly with a method of Harvey et al. [26].

2. if $\pi^{k,\ell}$ exists, let $\hat{\pi}^{k,\ell}$ be the allocation by allocating indivisible goods according to $\pi^{k,\ell}$, and allocating divisible goods by a water-filling policy so that

$$\hat{\pi}_i^{k,\ell}(c) = \frac{1}{|C|} \cdot \left( \beta_{j^*} - \frac{|N_{j^*}^+| \cdot \beta_{j^*} - \ell - |C|}{|N_{j^*}^+| - k} - \pi_i^{k,\ell}(M) \right)$$

for each $i \in N_{j^*}^+$ such that $\pi_i^{k,\ell}(M) < \beta_{j^*}$ and $c \in C$.

Let us see that this indeed works. Since $\Phi(\hat{\pi}^{k,\ell}) \leq \Phi(\hat{\pi}^{k+1,\ell})$ holds (if $\pi^{k,\ell}$ exists), the smallest $k$ such that $\pi^{k,\ell}$ exists is the number in Lemma 31. For such an integer $k$, we have $|\{i \in N_{j^*}^+ : \pi_i(M_{j^*}) = \beta_{j^*}\}| = k$, and the properties in Lemma 31 are satisfied except the allocation of divisible goods. Once we have decided on an allocation of indivisible goods, an optimal allocation of divisible goods is found by the water-filling policy.

Now, we explain how to find an allocation $\pi^{k,\ell}$ at step 1 in polynomial time. We reduce this problem to the submodular flow problem. Let $G = (V, A)$ be a directed graph constructed as follows. The set of vertices $V$ is $M_{j^*} \cup N_{j^*} \cup N'_{j^*}$ where $N'_{j^*}$ is a set of copy $i'$ of each $i \in N_{j^*}$. The set of edges $A$ is $A_1 \cup A_2 \cup A_3$ where $A_1 = \{(g, i') \in M_{j^*} \times N'_{j^*} : v_i(g) = 1\}$, $A_2 = \{(i', i) : i \in N_{j^*}^+\}$, and $A_3 = \{(i', i) : i \in N_{j^*}^-\}$. We define $\underline{c}, \bar{c} \colon A \to \mathbb{Z}$ as $\underline{c}(a) = 0$ and $\bar{c}(a) = 1$ for each $a \in A_1$; $\underline{c}(a) = 0$ and $\bar{c}(a) = \beta_{j^*}$ for each $a \in A_2$; $\underline{c}(a) = \beta_{j^*} - 1$ and $\bar{c}(a) = \beta_{j^*}$ for each $a \in A_3$. In addition, let $f_{k,\ell} \colon 2^V \to \mathbb{Z}$ be a function such that

$$f_{k,\ell}(X) = \varphi_{k,\ell}(|X \cap N_{j^*}^+|) + (|M_{j^*}| - \ell)\mathbf{1}_{X \cap N_{j^*}^- \neq \emptyset} - |X \cap M_{j^*}| \quad (\forall X \subseteq V), \tag{9}$$

where $\varphi_{k,\ell}(h) = \min\{\beta_{j^*}h,\ (\beta_{j^*} - 1)h + k,\ \ell\}$, and $\mathbf{1}_{X \cap N_{j^*}^- \neq \emptyset}$ takes the value 1 if $X \cap N_{j^*}^- \neq \emptyset$ and 0 otherwise. We remark that $f_{k,\ell}$ is a submodular function, and $f_{k,\ell}(V) = 0$ since $\ell \leq \sum_{i \in N_{j^*}^+} \pi_i^*(M_{j^*}) \leq (\beta_{j^*} - 1)|N_{j^*}^+| + k \leq \beta_{j^*}|N_{j^*}^+|$.

▶ **Lemma 32.** *There exists an allocation $\pi \in \{0,1\}^{N_{j^*} \times M_{j^*}}$ satisfying (a)–(d) if and only if there exists an integral flow $\xi \colon A \to \mathbb{Z}$ satisfying $\underline{c}(a) \leq \xi(a) \leq \bar{c}(a)$ (capacity constraints) and a constraint (called supply specification) that the boundary $\partial\xi \in \mathbb{Z}^V$ of the flow $\xi$, which is defined by $\partial\xi(v) = \sum_{a=(v,u) \in A} \xi(a) - \sum_{a=(u,v) \in A} \xi(a)$, is in the M-convex set $\ddot{\mathbf{B}}$ of $f_{k,\ell}$.*

Since the feasibility of the submodular flow problem can be determined in polynomial time [19], the existence of an allocation satisfying conditions (a)–(d) can be determined in polynomial time by Lemma 32. Moreover, if such an allocation exists, we can find one of such allocations in polynomial time.

Finally, because we do not know $\ell$ in advance, we enumerate all possibilities. That is, find a best allocation $\pi^{k,\ell}$ for each $\ell = 0, 1, \ldots, |M_{j^*}|$ by the above procedure, and choose the best one. Then the resulting allocation is as good as an optimal allocation $\pi^*$.

We give the formal description of our algorithm in Algorithm 1. By summarizing the discussions thus far, we can prove Theorem 4.

## 6 Hardness for Identical Indivisible Goods

In this section, we show a hardness result on the fair allocation setting when divisible goods are non-identical but indivisible goods are identical. By using Theorem 2, we prove the NP-hardness of finding a $\Phi$-fair allocation by using the 3-dimensional matching (3DM) problem, which is known to be NP-hard [21].

▶ **Theorem 33** (restatement of Theorem 3)**.** *For any fixed symmetric strictly convex function $\Phi$, the problem (1) is NP-hard even in the fair allocation setting with identical indivisible goods. Hence, finding a $\Phi$-fair allocation is NP-hard.*

**Algorithm 1** Allocation algorithm when the divisible goods are identical.

**input** : A fair allocation instance $(N, M, C, v)$ and a symmetric strictly convex function $\Phi$

**output** : A $\Phi$-fair allocation

Compute the canonical partition $N_1, \dots, N_q$, the essential values $\beta_1, \dots, \beta_q$, the canonical partition of the indivisible goods $M_1, \dots, M_q$, and the canonical partition of the divisible goods $C_1, \dots, C_q$;

Let $j^*$ be the index such that $C_{j^*} = C$;

**for** $j \leftarrow 1, \dots, j^* - 1, j^* + 1, \dots, q$ **do**

  Allocate $M_j$ to $N_j$ so that each agent receives $\beta_j$ or $\beta_j - 1$;

Let $N_{j^*}^+ \leftarrow \{j \in N_{j^*} : v_i(c) = 1 \ (\forall c \in C)\}$ and $N_{j^*}^- \leftarrow \{j \in N_{j^*} : v_i(c) = 0 \ (\forall c \in C)\}$;

Let $\Pi \leftarrow \emptyset$ be a set of candidate allocations;

**for** $k \leftarrow 0, 1, \dots, |N_{j^*}^+|$ *and* $\ell \leftarrow 0, 1, \dots, |M_{j^*}|$ **do**

  **if** $|N_{j^*}^+| \cdot (\beta_{j^*} - 1) + k \leq \ell + |C| \leq |N_{j^*}^+| \cdot \beta_{j^*}$ **then**

    Determine the existence an allocation $\pi^{k,\ell} \in \{0,1\}^{N_{j^*} \times M_{j^*}}$ satisfying the following conditions via the submodular flow problem:
    $|\{i \in N_{j^*}^+ : \pi_i(M_{j^*}) = \beta_{j^*}\}| \leq k$, $\pi_i(M_{j^*}) \leq \beta_{j^*}$ for each $i \in N_{j^*}^+$,
    $\sum_{i \in N_{j^*}^+} \pi_i(M_{j^*}) = \ell$, $\pi_i(M_{j^*}) \in \{\beta_{j^*}, \beta_{j^*} - 1\}$ for each $i \in N_{j^*}^-$;

    **if** *Such an allocation* $\pi^{k,\ell}$ *exists* **then**

      Let $\pi$ be an allocation such that indivisible goods are allocated according to Algorithm 1 and $\pi^{k,\ell}$, and the divisible goods are allocated to agents in $N_{j^*}^+$ by a water-filling policy;

      $\Pi \leftarrow \Pi \cup \{\pi\}$;

**return** $\pi^* \in \arg\min_{\pi \in \Pi} \Phi(\pi(E))$;

We can also prove the following from the same proof of this theorem.

▶ **Corollary 34.** *The problems of finding an MNW allocation and an optimal egalitarian allocation are both NP-hard, even when indivisible goods are identical.*

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